VERIFICATION OF BINOMIAL THEOREM AND CHU-VANDERMONDE CONVOLUTION BY THE FINITE DIFFERENCE METHOD

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Abstract. In this note, we show that Binomial theorem and Chu-Vandermonde convolution can both be verified by the finite difference method.

There are numerous binomial coefficient identities in the literature. Thereinto, Binomial theorem and Chu-Vandermonde convolution(cf. Bailey [1, ξ1.3]) can be stated, respectively, as

$$\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n, \tag{1}$$

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k}=\binom{x+y}{n}. \tag{2}$$

Let $\triangle$ be the usual difference operator with the unit increment. For a complex function $f(\tau)$, the finite difference of order n can be calculated through the following Newton-Gregory formula(cf. Graham et al. [3, ξ5.3]):

$$\triangle^n f(\tau)=\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}f(\tau+k). \tag{3}$$

When $f(\tau)$ is a polynomial of degree $m\leq n$, then $\triangle^n f(\tau)$ vanishes for $0\leq m<n$ and otherwise, equals $m!$ times the leading coefficient of $f(\tau)$ for $m=n$. Based on (1) and (2), Chu [2] derived respectively Abel's identities and Hagen-Rothe convolutions by the finite difference method. Inspired by the work just mentioned, we shall show that (1) and (2) can also be verified in the same way.

Verification of Binomial theorem:

Define the function $f(\tau)$ by

$$f(\tau)=(-x)^{\tau}.$$

On one hand, we obtain, according to (3), the relation

$$\triangle^n f(\tau)=\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}(-x)^{\tau+k}. \tag{4}$$

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On the other hand, it is not difficult to get the result

$$\triangle^n f(\tau) = (-x)^\tau (-x-1)^n. \tag{5}$$

Combing (4) with (5), we derive the equation

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} (-x)^{\tau+k} = (-x)^\tau (-x-1)^n.$$

Dividing both sides of the last equation by $(-1)^n(-x)^\tau$, we deduce (1) to complete the verification.

Verification of Chu-Vandermonde convolution:

It is easy to see that (2) is right when $n = 0$. We shall also assume $n > 0$ in the following argumentations. For a natural number i with $1 \le i \le n$, define the function $f(y)$ by

$$f(y) = \binom{y-i}{n-i}$$

which is a polynomial of degree $n-i < n$ in y. According to (3), we get the relation

$$\triangle^n f(y) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{y+k-i}{n-i} = 0. \tag{6}$$

Define the function $F(x)$ by

$$F(x) = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}$$

which is a polynomial of degree n in x. Considering that

$$F(i-y-1) = \sum_{k=0}^{n} \binom{i-y-1}{k} \binom{y}{n-k} = \frac{\binom{y}{i}}{\binom{n}{i}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{y+k-i}{n-i},$$

we asserts that (6) leads to $F(i-y-1) = 0$. Thereby, we find that all the zeros of $F(x)$ are given by $\{i-y-1 : 1 \le i \le n\}$. Observing further that the polynomial $\binom{x+y}{n}$ has the same zeros as $F(x)$, there must exist a constant θ such that

$$F(x) = \theta \binom{x+y}{n} \quad \text{with} \quad \theta = 1$$

where θ has been determined by letting $x = 0$ in the last equation. In conclusion, we have verified the correctness of (2).

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